NUMERICAL SOLUTION FOR NONISOTHERMAL MOTION OF A VISCOUS LIQUID IN A TWO-DIMENSIONAL TUBE

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The problem of simultaneous development of velocity and temperature profiles in a two-dimensional tube is examined. The assumption is made that the liquid viscosity depends on temperature, while the other parameters are constant.

Some problems concerning the motion of an incompressible liquid whose viscosity depends on temperature were examined in [1–4]. The present paper examines the stabilization of a uniform velocity profile. For isothermal flow, a numerical solution was given in [5] for the approximate equations describing the development of the motion in a two-dimensional tube.

Under the usual assumptions the equations for the development of velocity and temperature profiles in a two-dimensional tube have the form [1, 2]

$$\rho\left(v_x\frac{\partial v_x}{\partial x}+v_y\frac{\partial v_x}{\partial y}\right)=\frac{\partial}{\partial y}\left(\mu\frac{\partial v_x}{\partial y}\right)-\frac{\partial p}{\partial x},$$

$$\frac{\partial p}{\partial y}=0;$$

$$\rho c_p\left(v_x\frac{\partial T}{\partial x}+v_y\frac{\partial T}{\partial y}\right)=\lambda\frac{\partial^2 T}{\partial y^2}+\frac{\mu}{J}\left(\frac{\partial v_x}{\partial y}\right)^2,$$

$$\frac{\partial v_x}{\partial x}+\frac{\partial v_y}{\partial y}=0;$$

$$\mu=\mu(T) \qquad (1)$$

with the boundary conditions

$$v_x=V,\ T=T_1,\ p=p_0 \text{ when } x=0,$$

$$v_x=0,\ v_y=0,\ \alpha_1\frac{\partial T}{\partial y}+\alpha_2 T=A \text{ when } y=0,$$

$$v_x=0,\ v_y=0,\ \beta_1\frac{\partial T}{\partial y}+\beta_2 T=B \text{ when } y=h,$$

where h is the width of the tube, and V, T_1, α_1, α_2, A, β_1, β_2, B are constants.

Let us examine numerical solution of system (1), which would allow a solution to be found without further simplification. We introduce dimensionless variables according to

$$u=v_x/V,\quad v=\mathrm{Re}\,v_y/V,\quad \eta=y/h,\quad \xi=x/\mathrm{Re}\,h,$$

$$\vartheta=(T-T_1)T_0,\quad P=(p-p_0)/\rho v^2,$$

where $\mathrm{Re}=Vh\rho/\mu_0$; V, T_0, T_1 are characteristic constants.

We write

$$\mathrm{Pr}=c_p\mu_0/\lambda,\quad m=\mu_0V^2/\lambda T_0 J.$$

We assume the dependence of viscosity on the temperature of the liquid is expressed by $\mu=\mu_0\exp(-\alpha\vartheta)$ when α = const. System (1) may then be written in the form

$$u\frac{\partial u}{\partial\xi}+v\frac{\partial u}{\partial\eta}=\frac{\partial}{\partial\eta}\left[\exp(-\alpha\vartheta)\frac{\partial u}{\partial\eta}\right]-\frac{\partial P}{\partial\xi},$$

$$\frac{\partial u}{\partial\xi}+\frac{\partial v}{\partial\eta}=0, \qquad (2)$$

$$\mathrm{Pr}\left(u\frac{\partial\vartheta}{\partial\xi}+v\frac{\partial\vartheta}{\partial\eta}\right)=\frac{\partial^2\vartheta}{\partial\eta^2}+m\exp(-\alpha\vartheta)\left(\frac{\partial u}{\partial\eta}\right)^2$$

with boundary conditions

$$u=1,\ \vartheta=0,\ P=0 \text{ when } \xi=0,$$

$$u=0,\ v=0,\ \frac{\alpha_1}{h}\frac{\partial\vartheta}{\partial\eta}+\alpha_2\vartheta=\frac{A-\alpha_2T_1}{T_0} \text{ when } \eta=0,$$

$$u=0,\ v=0,\ \frac{\beta_1}{h}\frac{\partial\vartheta}{\partial\eta}+\beta_2\vartheta=\frac{B-\beta_2T_1}{T_0} \text{ when } \eta=1.$$

Replacing the derivatives by finite-difference relations, as is done in solving boundary layer equations, we obtain instead of (2) the finite-difference system

$$u_{i,k}\frac{u_{i+1,k}-u_{i,k}}{\Delta\xi}+v_{i,k}\frac{u_{i,k+1}-u_{i,k-1}}{2\Delta\eta}=$$

$$=\{\exp(-\alpha\vartheta_{i+1,k+1/2})u_{i+1,k+1}+\exp(-\alpha\vartheta_{i+1,k-1/2})u_{i+1,k-1}-$$

$$-[\exp(-\alpha\vartheta_{i+1,k+1/2})+$$

$$+\exp(-\alpha\vartheta_{i+1,k-1/2})]u_{i+1,k}\}\frac{1}{(\Delta\eta)^2}-\left(\frac{\partial P}{\partial\xi}\right)_{i+1},$$

$$v_{i+1,k}=v_{i+1,k-1}-\frac{\Delta\eta}{2\Delta\xi}\Big[u_{i+1,k-1}+u_{i+1,k}-$$

$$-(u_{i,k-1}+u_{i,k})\Big],$$

$$\mathrm{Pr}\left(u_{i,k}\frac{\vartheta_{i+1,k}-\vartheta_{i,k}}{\Delta\xi}+v_{i,k}\frac{\vartheta_{i,k+1}-\vartheta_{i,k-1}}{2\Delta\eta}\right)=$$

$$=\frac{\vartheta_{i+1,k+1}+\vartheta_{i+1,k-1}-2\vartheta_{i+1,k}}{(\Delta\eta)^2}+$$

$$+m\exp(-\alpha\vartheta_{i,k})\left(\frac{u_{i,k+1}-u_{i,k-1}}{2\Delta\eta}\right)^2 \qquad (3)$$

with appropriate boundary conditions. System (3) determines the solution at the points of the rectangular net $\xi_i=i\Delta\xi,\eta_k=k\Delta\eta$ $(i=1,2,3,\ldots;\ k=0,1,2,\ldots,K)$, where $\vartheta_{i,k\pm1/2}=(\vartheta_{i,k\pm1}+\vartheta_{i,k})/2$. For known values $u_{i,k}$, $v_{i,k}$, $\vartheta_{i,k}$ we find the quantities $\vartheta_{i+1,k}$ from the third

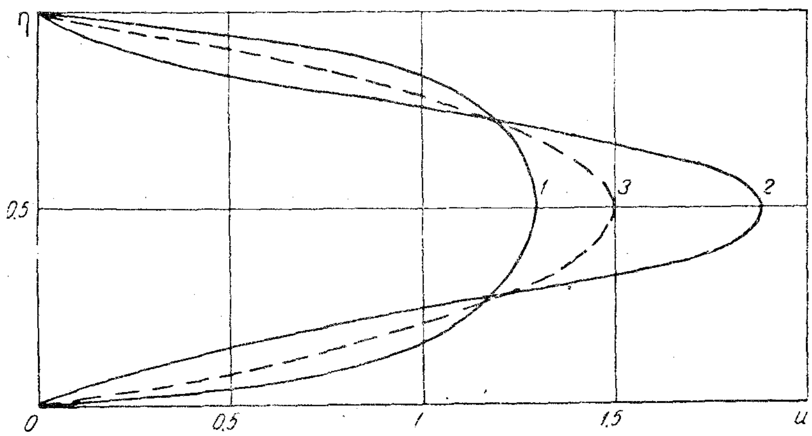

Fig. 1. Velocity profiles in a two-dimensional tube with $\xi = 0.05$, $Pr = 100$, $m = 0.1$. 1) $k = 1$, $\alpha = 2$; 2) $k = -1$, $\alpha = 2$; 3) $\alpha = 0$.

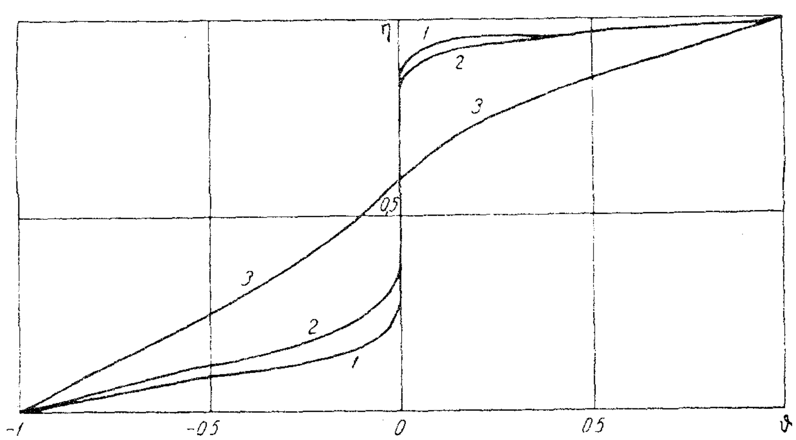

Fig. 2. Temperature profiles $\vartheta = (T - T_1)/T_1$ at various distances from the inlet, under asymmetric heating and $Pr = 100$, $\alpha = 2$, $m = 0$, $\xi = 0.0025$ (1); 0.05 (2); and 3.9 (3).

equation of system (3). Thereafter, the first equation of (3) may be put in the form

$$M_{i+1,k}u_{i+1,k+1}+N_{i+1,k}u_{i+1,k}+C_{i+1,k}u_{i+1,k-1}=D_{i+1,k}+E_{i+1,k}\left(\frac{\partial P}{\partial \xi}\right)_{i+1}. \quad (4)$$

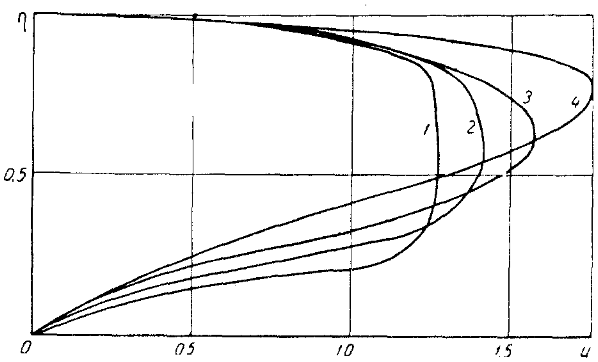

Fig. 3. Velocity profiles at different distances from the inlet with asymmetrical heating Pr = = 100, $\alpha = 2$, m = 0, and $\xi = 0.0025$ (1); 0.01 (2); 0.05 (3); 3.9 (4).

Values of the quantities $M_{i+1,k}$, $N_{i+1,k}$, $C_{i+1,k}$, $D_{i+1,k}$, $E_{i+1,k}$ are known. We seek a solution of (4) in the form

$$u_{i+1,k}=w_{i+1,k}+z_{i+1,k}\left(\frac{\partial P}{\partial \xi}\right)_{i+1}. \quad (5)$$

The quantities $w_{i+1,k}$ and $z_{i+1,k}$ satisfy the difference equations

$$M_{i+1,k}w_{i+1,k+1}+N_{i+1,k}w_{i+1,k}+C_{i+1,k}w_{i+1,k-1}=D_{i+1,k}, \quad (6)$$

$$M_{i+1,k}z_{i+1,k+1}+N_{i+1,k}z_{i+1,k}+C_{i+1,k}z_{i+1,k-1}=E_{i+1,k} \quad (7)$$

with zero boundary conditions. Integrating the second equation of (2) over the width of the tube, we obtain the condition for constant mass flowrate:

$$\int_0^1 u\,d\eta = q,$$

where q is a known constant. Replacing the integral in this condition by a finite sum according to the quadrature formula and using (5), we obtain the condition for determining the pressure derivative:

$$\left(\frac{\partial P}{\partial \xi}\right)_{i+1}=\left(q-\frac{1}{2}w_{i+1,0}+w_{i+1,1}+\dots+w_{i+1,K-1}+\frac{1}{2}w_{i+1,K}\right)\left(\frac{1}{2}z_{i+1,0}+z_{i+1,1}+\dots+z_{i+1,K-1}+\frac{1}{2}z_{i+1,K}\right)^{-1}. \quad (8)$$

The trapezoid formula was used in (8).

From (5) we find the quantity $u_{i+1,k}$ and from the second equation of (3) the quantity $v_{i+1,k}$. Thereafter we proceed to determine the quantities in the following row. The difference systems were solved by the pivotal condensation method. Calculations were done for constant liquid temperature at the inlet (T_1) and constant wall temperature (T_0), the quantity $k = (T_0 - T_1)/T_1$ assuming different values. Since the viscosity depends on temperature, the length of the section in which complete hydrodynamic stabilization occurs is the same as that for thermal stabilization, but in the case of isothermal flow when Pr > 1 it is greater than the length for thermal stabilization. Since when $\xi \to \infty$ the liquid assumes a temperature close to that of the walls (the dissipation effect at the values of the parameters chosen here is small), the velocity profiles, even for large values of the variable ξ, are close to parabolic.

The whole interval of velocity variation may be divided into two parts. In the first part (up to $\xi \approx \approx 0.05$) the velocity profile is observed to change rapidly from being uniform at the inlet section to some profile corresponding to different temperatures of flow core and layer near the wall. The length of this section is close to that for hydrodynamic stabilization in isothermal flow and does not alter much for various temperature conditions (Fig. 1). In the second section there occurs a simultaneous slow change of velocity and temperature profiles, terminating at the end of thermal stabilization.

At the usual values of the parameters, the effect of dissipation on the nature of the initial section proves to be small, and in some calculations it is assumed that m = 0. Calculations were carried out for values of the parameters for which there exists a stable solution limiting with respect to ξ [6].

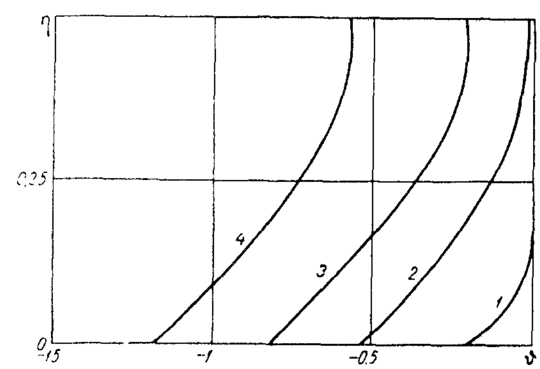

Fig. 4. Temperature profiles for constant heat flux through the wall, Pr = 100, $\alpha = 2$, m = 0, and $\xi = 0.1$ (1); 10 (2); 30 (3); 54.5 (4).

A calculation was also done for the case when the wall of a two-dimensional channel is maintained at various temperatures different from that of the liquid at the tube inlet, the temperature of one wall being above and that of the other being below the inlet temperature of the liquid. The velocity profile in this case changes from being uniform at the inlet to some asymmetrical profile, the asymmetry being determined by the dependence of viscosity on temperature in conformity with the change of temperature profile (Figs. 2 and 3). A calculation was also carried out for constant heat flux through the tube walls, given by the condition

$$\frac{\partial T}{\partial y} = Q \text{ when } y=0, \qquad \frac{\partial T}{\partial y} = 0 \text{ when } y = \frac{h}{2}.$$

In this case the dimensionless temperature is determined by the formula

$$\vartheta = (T - T_1)/Qh,$$

which leads to the boundary condition

$$\frac{\partial \vartheta}{\partial \eta} = 1 \text{ when } \eta = 0, \quad \frac{\partial \vartheta}{\partial \eta} = 0 \text{ when } \eta = 0.5.$$

The calculations were carried out for $Q > 0$, i.e., for uniform two-sided cooling of the liquid. Since the liquid temperature close to the inlet differs little from its value at the inlet, the velocity profile changes from being uniform to near-parabolic. Later on, a slow change of velocity profile takes place, in accordance with the change of temperature profile. At some distance from the inlet section the temperature profiles change, while remaining similar, but the velocity profiles at a given mass flowrate do not change (Fig. 4).

In the calculations steps $\Delta\xi$ and $\Delta\eta$ were subdivided until the results agreed to three significant figures. At $Pr \gg 1$ the third equation of (2) has a small parameter in the leading derivative. We shall examine the model equation

$$A \frac{\partial \vartheta}{\partial \xi} + B \frac{\partial \vartheta}{\partial \eta} = 0,$$

where A = const, B = const. In this case the scheme employed in the present paper is close to the scheme

$$\frac{\vartheta_{i+1,k} - \vartheta_{i,k}}{\Delta\xi} + \frac{B}{A}\frac{\vartheta_{i,k+1} - \vartheta_{i,k-1}}{2\Delta\eta} = 0,$$

which is stable when $B\Delta\xi/A(\Delta\eta)^2 = \text{const}$ [7].

When there is cooling of the liquid, the quantity v/u is large in the wall layer in the initial section, and when the value of $\Delta\eta$ is reduced, a considerable reduction in $\Delta\xi$ occurs (we note that in these conditions there is no justification for simplifying the original system of Navier-Stokes equations).

NOTATION

v_x, v_y—velocity components; p—pressure; T—temperature; μ—viscosity; J—mechanical equivalent of heat; h—width of tube; V—characteristic velocity; T_0, T_1—temperature of wall and of liquid at inlet; u, v—components of dimensionless velocity; P—dimensionless pressure; ϑ—dimensionless temperature; ξ, η—dimensionless coordinates; q—dimensionless mass flowrate; α_1, α_2, A, β_1, β_2, B—arbitrary constants; α—exponent in viscosity-temperature relation.

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